Cell algebra stuctures on monoid and twisted monoid algebras

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1 Introduction

In [11] a class of algebras called cell algebras is defined generalizing the cellular algebras of Graham and Lehrer in [7]. (These algebras had previously been introduced and studied as "standardly based algebras" by Du and Rui in [4].) It was shown in [11] that if R is any commutative domain with unit and M is either a transformation semigroup $\mathcal{T}_r$ or a partial transformation semigroup $\mathcal{PT}_r$ then the semigroup algebra $R[M]$ is a cell algebra with a standard cell basis. In this paper we show that the algebra $R[M]$ is a cell algebra for any finite monoid M with the property (called the "R-C.A." property) that for every $\mathcal{D}$-class D the group algebra $R[G_D]$ for the Schutzenberger group of D has a cell algebra structure. The choice of a cell basis for each $R[G_D]$ determines a standard cell basis for $R[M]$. If all the groups G_D for a given M are symmetric groups (as for $\mathcal{T}_r$ or $\mathcal{PT}_r$) then M satisfies the R-C.A. property for any R. If k is an algebraically closed field of characteristic 0 or p where p does not divide the order of any G_D, then any finite monoid M satisfies the k-C.A. property.

We use the standard cell basis obtained for M and the properties of cell algebras given in [11] to derive properties of the algebras $R[M]$ for such monoids. For example, if M is a regular semigroup and each cell algebra $R[G_D]$ has the property that $(\Lambda_D)_0 = \Lambda_D$ then $R[M]$ is quasi-hereditary. As a special case, this yields the theorem of Putcha [13] that the complex monoid algebra $\mathbb{C}[M]$ is quasi-hereditary for any finite regular monoid. If M is an inverse semigroup satisfying the R-C.A. property, then we show that $R[M]$ is semi-simple if and only if each cell algebra $R[G_D]$ is semi-simple. When R is an algebraically closed field k of good characteristic, this yields that $k[M]$ is semi-simple for any inverse semi-group (a result long known even without the algebraically closed condition; see [1] which cites [12]).

We also show that if $\pi : M \times M \to R$ is any twisting satisfying a certain simple compatibility condition then the twisted monoid algebra $R^\pi[M]$ is also a cell algebra (with the same sets Λ, L, R and cell basis C as the cell algebra $R[M]$).

In [5], [14], and [9], East, Wilcox, and Guo and Xi have explored analogous results for cellular algebra structures on semigroups and twisted semigroups. Many of the complications in their work arise from the need to construct the

involution anti-automorphism $*$ required by the definition of a cellular algebra. Cell algebras require no such mapping, and the results obtained for such algebras appear to be both considerably simpler and of more general applicability than the corresponding result for cellular algebras. Yet questions such as whether an algebra is quasi-hereditary or semi-simple appear to be not much harder to address when given a cell basis than when given a cellular basis.

2 Properties of finite monoids

In this section we review some facts about finite monoids and the standard Green's relations on a finite monoid M. Most of these results are well-known; see, for example, [1], [6], or [10] for references.

Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ are defined, for $x, y \in M$, by

$$x\mathcal{L}y \Leftrightarrow Mx = My\ ,\ x\mathcal{R}y \Leftrightarrow xM = yM\ ,\ x\mathcal{J}y \Leftrightarrow MxM = MyM$$

$$x\mathcal{H}y \Leftrightarrow x\mathcal{L}y \text{ and } x\mathcal{R}y$$

$$x\mathcal{D}y \Leftrightarrow x\mathcal{L}z \text{ and } z\mathcal{R}y \text{ for some } z \in M \Leftrightarrow x\mathcal{R}z \text{ and } z\mathcal{L}y \text{ for some } z \in M.$$

If $\mathcal{K}$ is one of Green's relations and $m \in M$, denote by K_m the $\mathcal{K}$-class containing m. Write $\mathbb{L}$, $\mathbb{R}$, $\mathbb{J}$, $\mathbb{H}$, or $\mathbb{D}$ for the set of all $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, or $\mathcal{D}$ classes in M.

According to "Green's lemma", if $a\mathcal{R}b$ and $s, t \in M$ are such that $as = b$, $bt = a$, then the map $x \to xs$ is a bijective, $\mathcal{R}$-class preserving map from L_a onto L_b with inverse the map $y \to yt$. There is a dual result when $a\mathcal{L}b$

For a finite monoid M, one has $\mathcal{D} = \mathcal{J}$ (see chapter 5 of [8]) and the set $\mathbb{D}$ of $\mathcal{D}$-classes inherits a partial order defined, for $x, y \in M$, by $D_x \leqslant D_y \Leftrightarrow x \in MyM$. Evidently $D_{xy} \leqslant D_x$ and $D_{xy} \leqslant D_y$ for any $x, y \in M$.

Let R be a commutative domain with unit. For any subset $S \subseteq M$, write $R[S]$ for the free R-module with basis S. Let $A = R[M]$, the monoid algebra for M, and define $A^D = \oplus_{D' \leqslant D} R[D']$, $\hat{A}^D = \oplus_{D' < D} R[D']$. Then both A^D and $\hat{A}^D$ are two sided ideals in A.

For a given $D \in \mathbb{D}$, choose a "base element" $\gamma \in D$ and consider the $\mathcal{H}$-class $H = H_\gamma \subseteq D$. Let $RT(H) = \{m \in M : Hm \subseteq H\}$ and for $m \in RT(H)$ define the right translation $r_m : H \to H$ by $(h)r_m = hm$, $h \in H$. Each map r_m is a bijection from H to H and $G_H^R = \{r_m : m \in RT(H)\}$ is a group, the (right) Schutzenberger group for H. (Up to isomorphism, G_H^R depends only on D, not the particular $\mathcal{H}$-class H contained in D.) Evidently $r_m r_n = r_{mn}$ for any $m, n \in RT(H)$. The map $\phi_H^R : G_H^R \to H$ defined by $\phi_H^R : r_m \mapsto \gamma_D m$ is bijective and extends linearly to a bijective map $\phi_H^R : R\left[G_H^R\right] \to R\left[H\right]$ of free R-modules. For any $m, n \in RT(H)$, $\phi_H^R(r_m) \cdot n = (\gamma_D m) \cdot n = \gamma_D(mn) = \phi_H^R(r_{mn}) = \phi_H^R(r_m r_n)$, so in general for any $x \in R\left[G_H^R\right]$ we have by linearity $\phi_H^R(x) \cdot n = \phi_H^R(x r_n)$.

In parallel fashion, let $LT(H) = \{m \in M : mH \subseteq H\}$ and for $m \in LT(H)$ define the left translation $l_m : H \to H$ by $l_m(h) = mh$, $h \in H$. Then again each map l_m is a bijection from H to H and $G_H^L = \{l_m : m \in LT(H)\}$ is a group, the (left, or dual) Schutzenberger group for H. Then $l_m l_n = l_{mn}$ for any $m, n \in LT(H)$, the map $\phi_H^L : G_H^L \to H$ defined by $\phi_H^L : l_m \mapsto m\gamma_D$ is bijective,

and the linearly extended map $\phi_H^L : R\left[G_H^L\right] \to R[H]$ is a bijection of free R-modules. The map $\psi = \left(\phi_H^R\right)^{-1} \circ \phi_H^L : G_H^L \to G_H^R$ is then also bijective. Choose an (injective) map $\vartheta : G_H^R \to RT(H)$ such that $r_{\vartheta(g)} = g$ for all $g \in G_H^R$. Then for any $m \in LT(H)$ define $\bar{m} \in RT(H)$ by $\bar{m} = \vartheta \circ \psi(l_m)$, so that $m\gamma_D = \gamma_D \bar{m}$. For any $m \in LT(H)\,,\, n \in RT(H)$ we have $m \cdot \phi_H^R(r_n) = m \cdot \gamma_D n = m\gamma_D \cdot n = \gamma_D \bar{m} \cdot n = \phi_H^R(r_{\bar{m}} r_n)$. Then for any $x \in R\left[G_H^R\right]$, $m \in LT(H)$ we have by linearity $m\phi_H^R(x) = \phi_H^R(r_{\bar{m}} x)$.

Lemma 2.1. *Take any $m \in M$.*

i. If there exists an $h \in H$ such that $hm \in H$ then $m \in RT(H)$ and r_m gives a bijective map of H onto H.

ii. If there exists an $h \in H$ such that $mh \in H$then $m \in LT(H)$ and l_m gives a bijective map of H onto H.

Proof. For i., write $h = m_1 \cdot \gamma_D$, $hm = m_2 \cdot \gamma_D$ for some $m_1, m_2 \in LT(H)$. Then $l_{m_1} \in G_H^L$ and we can choose $\bar{m}_1 \in LT(H)$ such that $l_{\bar{m}_1} = (l_{m_1})^{-1}$. Then $\gamma_D = \bar{m}_1 \cdot h$. Any element of H has the form $n \cdot \gamma_D$ for some $n \in LT(H)$, and we have $n\gamma_D \cdot m = n \cdot (\bar{m}_1 h) \cdot m = n\bar{m}_1(hm) = n\bar{m}_1(m_2\gamma_D) = (n\bar{m}_1 m_2) \cdot \gamma_D$. But $n\bar{m}_1 m_2 \in LT(H)$ (since $n, \bar{m}_1, m_2$ are), so $n\gamma_D \cdot m = (n\bar{m}_1 m_2) \cdot \gamma_D \in H$. Then $m \in RT(H)$ as desired. The proof of ii. is parallel. ☐

We now investigate general products am , ma where $a \in D \in \mathbb{D}$ and $m \in M$. As mentioned above, we have $D_{am} \leqslant D_a$ and $D_{ma} \leqslant D_a$. We need to study the cases when $D_{am} = D_a$ or $D_{ma} = D_a$. The first tool is the following result (see e.g chapter 5 of [8]):

Lemma 2.2. *For any finite monoid M and $a, m \in M$,*

i If $D_a = D_{am}$, then $R_a = R_{am}$.

ii If $D_a = D_{ma}$, then $L_a = L_{ma}$.

Proof. We use the facts (again, see chapter 5 of [8]) that in any finite monoid $\mathcal{D} = \mathcal{J}$ and that for any element x in a finite monoid some power x^n is idempotent. For i., assume $D_a = D_{am}$ for some $a, m \in M$. To prove $R_a = R_{am}$, it suffices to check that $a \in amM$ and $am \in aM$. Since $am \in aM$ is obvious, we need show only $a \in amM$. Now $D_a = D_{am} \Rightarrow J_a = J_{am} \Rightarrow a \in MamM \Rightarrow a = x(am)y = xa(my)$ for some $x, y \in M$. Then by repeated substitutions we have $a = x^k a(my)^k$ for any positive integer k. Choose $k = n$ so that x^n is an idempotent e. Then $a = ea(my)^n$ and $ea = e \cdot ea(my)^n = ea(my)^n = a$. But then $a = a(my)^n = am \cdot y(my)^{n-1} \in amM$ as desired. The proof of ii. is similar. ☐

We also use the "egg-box" picture of a given fixed class $D \in \mathbb{D}$:

Picture the $\mathcal{H}$-classes contained in D in a rectangular array where the $\mathcal{H}$-classes in a given column are all in the same $\mathcal{L}$-class and the $\mathcal{H}$-classes in a given row are all in the same $\mathcal{R}$-class. So the number of columns in the array is

the number $n(D,L)$ of distinct $\mathcal{L}$-classes in D, while the number of rows is the number $n(D,R)$ of distinct $\mathcal{R}$-classes in D. Write ${}_iH_j$ for the $\mathcal{H}$-class in row i, column j. We can assume that ${}_1H_1$ is our "base class" H_γ.

The following result follows from Green's lemma and its dual:

Lemma 2.3. *For each $i \in \{1, 2, \cdots, n(D,R)\}$, $j \in \{1, 2, \cdots, n(D,L)\}$ there exist elements $a_i, \bar{a}_i, b_j, \bar{b}_j \in M$ such that*

i. For each column j, the left translation $h \mapsto a_i h$ defines a bijective map $l_i : {}_1H_j \to {}_iH_j$ with inverse $l_i^{-1} : {}_iH_j \to {}_1H_j$ defined by left translation $l_i^{-1} : h \mapsto \bar{a}_i h$.

ii. For each row i, the right translation $h \mapsto hb_j$ defines a bijective map $r_j : {}_iH_1 \to {}_iH_j$ with inverse $r_j^{-1} : {}_iH_j \to {}_iH_1$ defined by right translation $r_j^{-1} : h \mapsto h\bar{b}_j$.

Then for any row i, column j, and $h \in H_\gamma$, $h \mapsto a_i h b_j$ gives a bijective map ${}_i\psi_j : H_\gamma = {}_1H_1 \to {}_iH_j$ of $\mathcal{H}$-classes which extends linearly to a bijection ${}_i\psi_j : R[H_\gamma] \to R[{}_iH_j]$.

We can now state the main result of this section.

Proposition 2.1. *Suppose that for some $d \in D, m \in M$ we have $dm \in D$. For some i,j, this $d \in {}_iH_j = {}_i\psi_j\,(H_\gamma) = a_i H_\gamma b_j$ and so $d = a_i \phi_H^R(g_d) b_j$ for some $g_d \in G_H^R$. Then:*

i. $dm \in {}_iH_k$ for some k. Define $m^* = b_j m \bar{b}_k$.

ii. $m^* \equiv b_j m \bar{b}_k \in RT\,(H)$.

iii. The right translation $r_m : h \mapsto hm$ gives a bijection $r_m : {}_iH_j \to {}_iH_k$.

iv. If $h = a_i \phi_H^R(g) b_j \in {}_iH_j$, then $hm = (h)r_m = a_i \phi_H^R\,(g\, r_{m^*})\, b_k$.

v. For any $x \in R\,[{}_iH_j]$, write $x = a_i \phi_H^R\,(y)\, b_j$ for some $y \in R\left[G_H^R\right]$. Then $xm = a_i \phi_H^R\,(y\, r_{m^*})\, b_k$.

Proof. Since $D_d = D_{dm} = D$, we have $R_d = R_{dm}$ by lemma 2.2. So H_d and H_{dm} lie in the same $\mathcal{R}$-class and are in the same row of the "egg-box" for D. So if $d \in {}_iH_j$, then $dm \in {}_iH_k$ for some k, proving i. Write $d = a_i \phi_H^R(g_d) b_j$ and $dm = a_i \phi_H^R(g_{dm}) b_k$ for some $g_d, g_{dm} \in G_H^R$. Then $dm = a_i \phi_H^R(g_d) b_j m = a_i \phi_H^R(g_{dm}) b_k$. Multiplying by $\bar{a}_i$ on the left and by $\bar{b}_k$ on the left gives $\phi_H^R(g_d) b_j m \bar{b}_k = \phi_H^R(g_{dm}) \in H$. So by lemma 2.1, $m^* \equiv b_j m \bar{b}_k \in RT\,(H)$ proving ii. Then $r_{m^*} : H \to H$ is a bijection $\phi_H^R\,(g)\, m^* = \phi_H^R\,(g r_{m^*})$ for any $g \in G_H^R$. Now take any $h = a_i \phi_H^R(g) b_j \in {}_iH_j$ and compute $hm\bar{b}_k = a_i \phi_H^R(g) b_j m \bar{b}_k = a_i \phi_H^R(g) m^* = a_i \phi_H^R(g r_{m^*})$ Multiplying on the right by b_k then proves iv., and v. then follows by linearity. For iii., observe that r_m can be written as a composition of bijections: $r_m = {}_i\psi_k \circ \phi_H^R \circ \bar{r}_{m^*} \circ \left(\phi_H^R\right)^{-1} \circ ({}_i\psi_j)^{-1}$ where $\bar{r}_{m^*} : G_H^R \to G_H^R$ is the bijection $g \mapsto g r_{m^*}$. ∎

As an immediate corollary we have

Corollary 2.1. *For any $\mathcal{H}$-class ${}_iH_j \subseteq D$ and any $m \in M$ either*

i. $R\left[{}_iH_j\right] \cdot m \subseteq \hat{A}^D = \oplus_{D'<D} R\left[D'\right]$ *or*

ii. $R\left[{}_iH_j\right] \cdot m = R\left[{}_iH_k\right]$ *for some* k *and if* $x = a_i\phi_H^R\left(y\right)b_j$ *for some* $y \in R\left[G_H^R\right]$*, then* $xm = a_i\phi_H^R\left(y\, r_{m^*}\right)b_k$*, where* $m^* = b_j m \bar{b}_k \in RT(H)$.

We obtain the dual versions of proposition 2.1 and corollary 2.1 given next by using part ii. of lemmas 2.1 and 2.2 along with the formula $m\phi_H^R(g) = \phi_H^R\left(r_{\bar{m}}g\right)$ for $g \in G_H^R$, where $\bar{m} = \vartheta \circ \psi\left(l_m\right) \in RT(H)$.

Proposition 2.2. *Suppose that for some $d \in D, m \in M$ we have $md \in D$. For some i,j, this $d \in {}_iH_j = {}_i\psi_j\left(H_\gamma\right) = a_iH_\gamma b_j$ and so $d = a_i\phi_H^R(g_d)b_j$ for some $g_d \in G_H^R$. Then:*

i. $md \in {}_kH_j$ *for some* k*. Define* $m^* = \bar{a}_k m a_j$.

ii. $m^* \equiv \bar{a}_k m a_j \in LT\left(H\right)$*, so* $\bar{m^*} = \vartheta \circ \psi\left(l_{m^*}\right) \in RT(H)$.

iii. The left translation $l_m : h \mapsto mh$ *gives a bijection* $l_m : {}_iH_j \to {}_kH_j$.

iv. If $h = a_i\phi_H^R(g)b_j \in {}_iH_j$*, then* $mh = l_m(h) = a_k\phi_H^R\left(r_{\bar{m^*}}\, g\right)b_j$.

v. For any $x \in R\left[{}_iH_j\right]$*, write* $x = a_i\phi_H^R\left(y\right)b_j$ *for some* $y \in R\left[G_H^R\right]$*. Then* $mx = a_k\phi_H^R\left(r_{\bar{m^*}}\ y\right)b_j$.

Corollary 2.2. *For any $\mathcal{H}$-class ${}_iH_j \subseteq D$ and any $m \in M$ either*

i. $m \cdot R\left[{}_iH_j\right] \subseteq \hat{A}^D = \oplus_{D'<D} R\left[D'\right]$ *or*

ii. $m \cdot R\left[{}_iH_j\right] = R\left[{}_kH_j\right]$ *for some* k *and if* $x = a_i\phi_H^R\left(y\right)b_j$ *for some* $y \in R\left[G_H^R\right]$*, then* $mx = a_k\phi_H^R\left(r_{\bar{m^*}}\ y\right)b_j$*, where* $m^* \equiv \bar{a}_k m a_j \in LT\left(H\right)$ *and* $\bar{m^*} = \vartheta \circ \psi\left(l_{m^*}\right) \in RT(H)$.

3 Cell algebra structures on monoid algebras

In [11], a class of algebras called cell algebras is defined which generalize the cellular algebras of Graham and Lehrer [7]. These algebras had previously been introduced and studied as "standardly based algebras" by Du and Rui in [4]. Such algebras share many of the nice properties of cellular algebras. We will give conditions on a monoid M and domain R such that the monoid algebra $R[M]$ will be a cell algebra and will construct a standard cell basis for such algebras. We first review the definition.

Let R be a commutative integral domain with unit 1 and let A be an associative, unital R-algebra. Let Λ be a finite set with a partial order $\geqslant$ and for each $\lambda \in \Lambda$ let $L\left(\lambda\right), R\left(\lambda\right)$ be finite sets of "left indices" and "right indices". Assume that for each $\lambda \in \Lambda, s \in L\left(\lambda\right)$, and $t \in R\left(\lambda\right)$ there is an element ${}_sC_t^\lambda \in A$ such that the map $\left(\lambda, s, t\right) \mapsto {}_sC_t^\lambda$ is injective and $C = \left\{{}_sC_t^\lambda : \lambda \in \Lambda, s \in L(\lambda), t \in R\left(\lambda\right)\right\}$ is a free R-basis for A. Define R-submodules of A by $A^\lambda = R$ - span of $\left\{{}_sC_t^\mu : \mu \in \Lambda, \mu \geqslant \lambda, s \in L(\mu), t \in R\left(\mu\right)\right\}$ and $\hat{A}^\lambda = R$ - span of $\left\{{}_sC_t^\mu : \mu \in \Lambda, \mu > \lambda, s \in L(\mu), t \in R\left(\mu\right)\right\}$.

Definition 3.1. *Given (A, Λ, C), A is a cell algebra with poset Λ and cell basis C if*

i *For any $a \in A, \lambda \in \Lambda,$ and $s, s' \in L(\lambda)$, there exists $r_L = r_L(a, \lambda, s, s') \in R$ such that, for any $t \in R(\lambda)$, $a \cdot {}_sC_t^\lambda = \sum_{s' \in L(\lambda)} r_L \cdot {}_{s'}C_t^\lambda \mod \hat{A}^\lambda$, and*

ii *For any $a \in A, \lambda \in \Lambda,$ and $t, t' \in R(\lambda)$, there exists $r_R = r_R(a, \lambda, t, t') \in R$ such that, for any $s \in L(\lambda)$, ${}_sC_t^\lambda \cdot a = \sum_{t' \in R(\lambda)} r_R \cdot {}_sC_{t'}^\lambda \mod \hat{A}^\lambda$.*

Now for each $\mathcal{D}$-class D in a finite monoid M, choose a base element γ_D and base $\mathcal{H}$-class $H = H_\gamma$ as above. Then define the Schutzenberger group G_D of D to be $G_D = G_H^R$. (Up to isomorphism, this is independent of the choice of base class H.) Let $R[G_D]$ be the group algebra of G_D over R.

Definition 3.2. *A monoid M satifies the R-C.A. condition for a given domain R if $R[G_D]$ has a cell algebra structure for every $\mathcal{D}$-class D in M.*

If an $\mathcal{H}$-class H contains an idempotent, then H is actually a subgroup of M. In fact, the maximal subgroups of M are just the $\mathcal{H}$-classes which contain an idempotent. In this case the group H is isomophic to the Schutzenberger group G_H^R. If M is a *regular* semigroup, then every $\mathcal{D}$-class D contains an idempotent e, so G_D is isomorphic to a maximal subgroup H_e of M. Then for regular semigroups M the R-C.A. condition is equivalent to requiring that $R[G]$ have a cell algebra structure for every maximal subgroup G of M.

If M is a monoid (such as a transformation semigroup $\mathcal{T}_r$ or a partial transformation semigroup $\mathcal{PT}_r$) for which every group G_D is a symmetric group, then the usual Murphy basis gives a cellular (and hence cell) algebra structure to $R[G_D]$ for any domain R. Thus M satisfies the R-C.A. condition for any R.

For any finite monoid M, if k is a field of characteristic 0 or characteristic p where p does not divide the order of any G_D, then by Maschke's theorem, every $k[G_D]$ is semisimple. If, in addition, k is algebraically closed, then each $k[G_D]$ is split semisimple. Such algebras are products of matrix algebras over k and have a natural cellular (hence cell) algebra basis. Thus if k is algebraically closed and of good characteristic relative to M, then any M satisfies the k-C.A. condition.

Our main result is the following theorem.

Theorem 3.1. *Let M be a finite monoid satifying the R-C.A. condition for a domain R. Then $A = R[M]$ is a cell algebra. The choice of a cell basis for each algebra $R[G_D]$ gives rise to a standard cell basis for A.*

Proof. For a given $D \in \mathbb{D}$, put $A_D = R[G_D]$ and assume Λ_D, L_D, R_D define a cell algebra structure on A_D with cell basis

$$C_D = \left\{ {}_sC_t^\lambda : \lambda \in \Lambda_D, s \in L_D(\lambda), t \in R_D(\lambda) \right\}.$$

Define a poset Λ to consist of all pairs (D, λ) where D is a $\mathcal{D}$-class in M and $\lambda \in \Lambda_D$. Define the partial order by $(D_1, \lambda_1) > (D_2, \lambda_2)$ if $D_1 < D_2$ or $D_1 =$

D_2 and $\lambda_1 > \lambda_2$ in Λ_{D_1}. For $(D,\lambda) \in \Lambda$, define $L(D,\lambda)$ to be all pairs (R,s) where R is an $\mathcal{R}$-class contained in D and $s \in L_D(\lambda)$. Similarly, define $R(D,\lambda)$ to be all pairs (L,t) where L is an $\mathcal{L}$-class contained in D and $t \in R_D(\lambda)$. Finally, given $(D,\lambda) \in \Lambda$, $(R,s) \in L(D,\lambda)$, $(L,t) \in R(D,\lambda)$, assume R corresponds to row i and L corresponds to column j in the "egg-box" for D. Then define

$$_{(R,s)}C_{(L,t)}^{(D,\lambda)} = {}_i\psi_j\left(\phi_H^R\left({}_sC_t^\lambda\right)\right) = a_i\phi_H^R\left({}_sC_t^\lambda\right)b_j.$$

For fixed $D,R,L,$ since $\left\{{}_sC_t^\lambda : \lambda \in \Lambda_D, s \in L_D(\lambda), t \in R_D(\lambda)\right\}$ is a basis for $R[G_D]$ by assumption and ${}_i\psi_j$ and ϕ_H^R are bijective,

$$\left\{{}_{(R,s)}C_{(L,t)}^{(D,\lambda)} : \lambda \in \Lambda_D, s \in L_D(\lambda), t \in R_D(\lambda)\right\}$$

will give a basis for $R[{}_iH_j]$. Then since $A = R[M]$ is the direct sum of the free submodules $R[H]$ as H varies over all $\mathcal{H}$-classes in M,

$$C = \left\{{}_{(R,s)}C_{(L,t)}^{(D,\lambda)} : (D,\lambda) \in \Lambda, (R,s) \in L(D,\lambda), (L,t) \in R(D,\lambda)\right\}$$

is a basis for A. To show C is a cell basis, we must check the cell conditions (i) and (ii). Note first that any element m in a $\mathcal{D}$-class D_m will be in the span of $\left\{{}_{(R,s)}C_{(L,t)}^{(D_m,\mu)} : \mu \in \Lambda_{D_m}, (R,s) \in L(D_m,\mu), (L,t) \in R(D_m,\mu)\right\}$. Then if $D_m < D$ for some $\mathcal{D}$-class D, we have $(D_m,\mu) > (D,\lambda)$ for any $\mu \in \Lambda_{D_m}$ and $\lambda \in \Lambda_D$, so $m \in \hat{A}^{(D,\lambda)}$. So $\oplus_{D'<D} R[D'] = \hat{A}^D \subseteq \hat{A}^{(D,\lambda)}$. To prove (i), we can assume a is a basis element $m \in M$. Take $(D,\lambda) \in \Lambda$, $(R,s) \in L(D,\lambda)$, $(L,t) \in R(D,\lambda)$ and assume R corresponds to row i and L corresponds to column j in the "egg-box" for D. Then ${}_{(R,s)}C_{(L,t)}^{(D,\lambda)} = {}_i\psi_j\left(\phi_H^R\left({}_sC_t^\lambda\right)\right) = a_i\phi_H^R\left({}_sC_t^\lambda\right)b_j \in R[{}_iH_j]$. By corollary 2.2 there are two cases to consider.

Case i: $m \cdot R[{}_iH_j] \subseteq \oplus_{D'<D} R[D']$. Then $m \cdot_{(R,s)} C_{(L,t)}^{(D,\lambda)} \in m \cdot R[{}_iH_j] \subseteq \oplus_{D<D'} R[D'] \subseteq \hat{A}^{(D,\lambda)}$ and we can satisfy (i) by taking all coefficients r_L to be zero.

Case ii: $m \cdot R[{}_iH_j] = R[{}_kH_j]$ for some k. By corollary 2.2, since ${}_{(R,s)}C_{(L,t)}^{(D,\lambda)} = a_i\phi_H^R\left({}_sC_t^\lambda\right)b_j \in R[{}_iH_j]$, then $m\ {}_{(R,s)}C_{(L,t)}^{(D,\lambda)} = a_k\phi_H^R\left(r_{\bar{m^*}}\,{}_sC_t^\lambda\right)b_j$, where $m^* \equiv \bar{a}_kma_j \in LT(H)$ and $\bar{m^*} = \vartheta \circ \psi(l_{m^*}) \in RT(H)$. Since $g = r_{\bar{m^*}} \in G_D$, the cell algebra property (i) for $R[G_D]$ gives $g \cdot {}_sC_t^\lambda = \sum_{s' \in L_D(\lambda)} r_L \cdot {}_{s'}C_t^\lambda \bmod \hat{A}_D^\lambda$, where r_L depends on $s, s', \lambda,$ and m, but is independent of t. Then

$$\begin{aligned} m\ {}_{(R,s)}C_{(L,t)}^{(D,\lambda)} &= a_k\phi_H^R\left(g\,{}_sC_t^\lambda\right)b_j \\ &= \sum_{s' \in L_D(\lambda)} r_L \cdot a_k\phi_H^R\left({}_{s'}C_t^\lambda\right)b_j \text{ modulo } a_k\phi_H^R\left(\hat{A}_D^\lambda\right)b_j \end{aligned}$$

where

$$a_k\phi_H^R\left(\hat{A}_D^\lambda\right)b_j \subseteq \operatorname{span}\left\{{}_{(R,s)}C_{(L,t)}^{(D,\mu)} : \mu > \lambda, s \in L_D(\mu), t \in R_D(\mu)\right\} \subseteq \hat{A}^{(D,\lambda)}.$$

But $a_k \phi_H^R \left({}_{s'}C_t^\lambda\right) b_j = {}_{(R',s')}C_{(L,t)}^{(D,\lambda)}$ where R' is the $\mathcal{R}$-class corresponding to row k of the "egg-box". Then $m \cdot {}_{(R,s)}C_{(L,t)}^{(D,\lambda)} = \sum_{s' \in L_D(\lambda)} r_L \cdot a_k \phi_H^R \left({}_{s'}C_t^\lambda\right) b_j = \sum_{s' \in L_D(\lambda)} r_L \cdot {}_{(R',s')}C_{(L,t)}^{(D,\lambda)} \mod \hat{A}^{(D,\lambda)}$. Since r_L is independent of L and t, this yields property (i) for this case ii.

The proof of condition (ii) is parallel. Thus C is a cell basis and $A = R[M]$ is a cell algebra. □

If M is a finite monoid satisfying the R-C.A. condition, we will assume a fixed cell algebra structure is given to each $R[G_D]$. We will then call the cell algebra structure obtained in the proof of theorem 3.1 the *standard cell algebra structure* on $R[M]$.

In [5], [14], and [9], East, Wilcox, and Guo and Xi worked with finite regular semigroups with cellular algebra (hence cell algebra) structure on $R[G]$ for maximal subgroups G of M. Their examples therefore satisfy the R-C.A. condition and can be seen to be cell algebras without considering the complicated involution requirements involved in showing a cellular algebra structure. In [5], East gives examples of inverse semigroups with cellular $R[G]$ for all maximal G which lack an appropriate involution and therefore do not have cellular $R[M]$. These examples would be cell algebras by theorem 3.1.

More typical examples of cell algebras that are not cellular are the algebras $R[M]$ for M a transformation semigroup $\mathcal{T}_r$ or partial transformation semigroup $\mathcal{PT}_r$. These were shown to be cell algebras in [11]. For these examples, a $\mathcal{D}$-class D consists of mappings of a given rank i and the group G_D is the symmetric group $\mathfrak{G}_i$. Since, as remarked above, the symmetric groups have cellular (hence cell) structures on their group algebras, theorem 3.1 applies and provides a cell algebra structure on M.

4 Properties of the cell algebra $A = R[M]$

In this section we assume that M is a finite monoid satisfying the R-C.A. condition, that is, such that for every $D \in \mathbb{D}$ the group algebra $R[G_D]$ of the Schutzenberger group for D is a cell algebra. Then by Theorem 3.1, $A = R[M]$ is a cell algebra and we can apply the results in [11] to A with the standard cell algebra structure.

For $(D,\lambda) \in \Lambda$, the left cell module ${}_LC^{(D,\lambda)}$ is a left A-module which is a free R-module with basis $\left\{{}_{(R,s)}C^{(D,\lambda)} : (R,s) \in L(D,\lambda)\right\}$. Similarly, the right cell module $C_R^{(D,\lambda)}$ for (D,λ) is a right A-module and a free R-module with basis $\left\{C_{(L,t)}^{(D,\lambda)} : (L,t) \in R(D,\lambda)\right\}$. For each $(D,\lambda) \in \Lambda$ there is an R-bilinear map $\langle -,- \rangle : \left(C_R^{(D,\lambda)},\ {}_LC^{(D,\lambda)}\right) \to R$ defined by the property $\left({}_{(R',s')}C_{(L,t)}^{(D,\lambda)}\right) \cdot \left({}_{(R,s)}C_{(L',t')}^{(D,\lambda)}\right) = \left\langle C_{(L,t)}^{(D,\lambda)},\ {}_{(R,s)}C^{(D,\lambda)}\right\rangle\ {}_{(R',s')}C_{(L',t')}^{(D,\lambda)} \mod \hat{A}^{(D,\lambda)}$ for any choice of R', s', L', t'.

Right and left radicals are defined by

$$\operatorname{rad}\left(C_R^{(D,\lambda)}\right) = \left\{x \in C_R^{(D,\lambda)} : \langle x, y\rangle = 0 \text{ for all } y \in {}_LC^{(D,\lambda)}\right\}$$

$$\operatorname{rad}\left({}_LC^{(D,\lambda)}\right) = \left\{y \in {}_LC^{(D,\lambda)} : \langle x, y\rangle = 0 \text{ for all } x \in C_R^{(D,\lambda)}\right\}.$$

Then define $D_R^{(D,\lambda)} = \frac{C_R^{(D,\lambda)}}{\operatorname{rad}\left(C_R^{(D,\lambda)}\right)}$ and ${}_LD^{(D,\lambda)} = \frac{{}_LC^{(D,\lambda)}}{\operatorname{rad}\left({}_LC^{(D,\lambda)}\right)}$. Finally, define $\Lambda_0 = \left\{(D,\lambda) \in \Lambda : \langle x, y\rangle \neq 0 \text{ for some } x \in C_R^{(D,\lambda)}, y \in {}_LC^{(D,\lambda)}\right\}$. Evidently, $\lambda \in \Lambda_0 \Leftrightarrow D_R^{(D,\lambda)} \neq 0 \Leftrightarrow {}_LD^{(D,\lambda)} \neq 0$. A major result of [11] is

Theorem 4.1. *Assume $R = k$ is a field. Then*
(a) $\left\{D_R^{(D,\mu)} : (D,\mu) \in \Lambda_0\right\}$ is a complete set of pairwise inequivalent irreducible right A-modules and
(b) $\left\{{}_LD^{(D,\mu)} : (D,\mu) \in \Lambda_0\right\}$ is a complete set of pairwise inequivalent irreducible left A-modules.

To find Λ_0 for A we need to relate the bracket $\langle -, -\rangle$ on A to the brackets $\langle -, -\rangle_D$ on the cell algebras $A_D = R[G_D]$. For a given $\mathcal{D}$-class D, write R_i for the $\mathcal{R}$-class corresponding to row i of the "egg-box" and L_j for the $\mathcal{L}$-class corresponding to column j.

Definition 4.1. *R_i, L_j are matched if there exist $x \in L_j$, $y \in R_i$ such that $xy \in D$. R_i, L_j are unmatched if $L_jR_i \cap D = \emptyset$.*

Lemma 4.1. *Assume R_i, L_j are matched. Write $x = a_{i'}\phi_H^R(\xi)b_j \in R[L_j]$ and $y = a_i\phi_H^R(\eta)b_{j'} \in R[R_i]$ for any $\xi, \eta \in R[G_D]$. Then $xy = a_{i'}\phi_H^R\left(\xi r_{m(i,j)}\eta\right)b_{j'}$ where $m(i,j) \equiv b_ja_i\gamma \in RT(H)$.*

Proof. If R_i, L_j are matched, then for some choice of i', j' and some elements $x \in {}_{i'}H_j \subseteq L_j$, $y \in {}_iH_{j'} \subseteq R_i$ we have $xy \in {}_{i'}H_{j'} \subseteq D$. But then (for the given i', j') we have $xy \in {}_{i'}H_{j'} \subseteq D$ for *any* $x \in {}_{i'}H_j$, $y \in {}_iH_{j'}$ by propositions 2.1 and 2.2. In particular, for $x = a_{i'}\gamma b_j \in {}_{i'}H_j$, $y = a_i\gamma b_{j'} \in {}_iH_{j'}$ we get $xy = a_{i'}\gamma b_ja_i\gamma b_{j'} \in {}_{i'}H_{j'}$. Then multiplying by $\bar{a}_{i'}$ on the left and $\bar{b}_{j'}$ on the right gives $\gamma b_ja_i\gamma \in H$. But then $m(i,j) \equiv b_ja_i\gamma \in RT(H)$ by lemma 2.1.

Now consider any i', j' and any $g = r_m, h = r_n \in G_D$ and let

$$x = a_{i'}\phi_H^R(g)b_j \in R[L_j]\,,\; y = a_i\phi_H^R(h)b_{j'} \in R[R_i].$$

Then

$$\begin{aligned} xy &= (a_{i'}\phi_H^R(g)\,b_j)(a_i\phi_H^R(h)\,b_{j'}) = a_{i'}\gamma m b_j a_i \gamma n b_{j'} \\ &= a_{i'}\gamma m \cdot m(i,j) \cdot n b_{j'} = a_{i'}\phi_H^R\left(r_m r_{m(i,j)} r_n\right)b_{j'} \\ &= a_{i'}\phi_H^R\left(g r_{m(i,j)} h\right)b_{j'}. \end{aligned}$$

Then by linearity of ϕ_H^R, $xy = a_{i'}\phi_H^R\left(\xi r_{m(i,j)}\eta\right)b_{j'}$ for arbitrary $\xi, \eta \in R[G_D]$ when $x = a_{i'}\phi_H^R(\xi)b_j \in R[L_j]$, $y = a_i\phi_H^R(\eta)b_{j'} \in R[R_i]$, proving the lemma. $\square$

For $i \in \{1, 2, \cdots, n(D,R)\}$, define $\left({}_LC^{(D,\lambda)}\right)_i$ to be the free R-module with basis $\left\{{}_{(R_i,s)}C^{(D,\lambda)} : \lambda \in \Lambda_D, s \in L_D(\lambda)\right\}$, so ${}_LC^{(D,\lambda)} = \bigoplus_i \left({}_LC^{(D,\lambda)}\right)_i$. Similarly, for $j \in \{1, 2, \cdots, n(D,L)\}$, define $\left(C_R^{(D,\lambda)}\right)_j$ to be the free R-module with basis $\left\{C^{(D,\lambda)}_{(L_j,t)} : \lambda \in \Lambda_D, t \in R_D(\lambda)\right\}$, so $C_R^{(D,\lambda)} = \bigoplus_j \left(C_R^{(D,\lambda)}\right)_j$. Notice that for each i, $\phi_i : {}_{(R_i,s)}C^{(D,\lambda)} \mapsto {}_sC^\lambda$ gives an isomorphism (of R-modules) $\phi_i : \left({}_LC^{(D,\lambda)}\right)_i \to {}_LC^\lambda$. Similarly, $\phi_j : C^{(D,\lambda)}_{(L_j,t)} \mapsto C_t^\lambda$ gives an isomorphism $\phi_j : \left(C_R^{(D,\lambda)}\right)_j \to C_R^\lambda$.

Proposition 4.1. *Take* $X \in \left(C_R^{(D,\lambda)}\right)_j$, $Y \in \left({}_LC^{(D,\lambda)}\right)_i$. *Then*

(a) *If* R_i , L_j *are not matched, then* $\langle X, Y\rangle = 0$,

(b) *If* R_i , L_j *are matched, then*

$$\langle X, Y\rangle = \left\langle \phi_j(X), r_{m(i,j)}\phi_i\left(Y\right)\right\rangle_D = \left\langle r_{m(i,j)}\phi_j(X), \phi_i\left(Y\right)\right\rangle_D$$

where $m(i,j) \equiv b_j a_i \gamma \in RT(H)$.

Proof. It suffices to check (a) and (b) for basis elements X, Y, so assume $X = C^{(D,\lambda)}_{(L_j,t)}$, $Y = {}_{(R_i,s)}C^{(D,\lambda)}$ and put $x = {}_{(R_{i'},s')}C^{(D,\lambda)}_{(L_j,t)} = a_{i'}\phi_H^R\left({}_{s'}C_t^\lambda\right)b_j \in L_j$ and $y = {}_{(R_i,s)}C^{(D,\lambda)}_{(L_{j'},t')} = a_i\phi_H^R\left({}_sC_{t'}^\lambda\right)b_{j'} \in R_i$. Then $xy = \langle X, Y\rangle \; {}_{(R_{i'},s')}C^{(D,\lambda)}_{(L_{j'},t')}$ mod $\hat{A}^{(D,\lambda)}$.

If R_i , L_j are not matched, then $xy \in \hat{A}^D \subseteq \hat{A}^{(D,\lambda)}$, so $\langle X, Y\rangle = 0$, proving (a).

If R_i , L_j are matched, then, by lemma 4.1,

$$\begin{aligned}
xy &= a_{i'}\phi_H^R\left({}_{s'}C_t^\lambda \cdot r_{m(i,j)} \cdot {}_sC_{t'}^\lambda\right)b_{j'} \\
&= a_{i'}\phi_H^R\left(\left\langle C_t^\lambda, r_{m(i,j)} \cdot {}_sC^\lambda\right\rangle_D {}_{s'}C_{t'}^\lambda\right)b_{j'} \bmod \hat{A}^{(D,\lambda)} \\
&= \left\langle C_t^\lambda, r_{m(i,j)} \cdot {}_sC^\lambda\right\rangle_D \cdot a_{i'}\phi_H^R\left({}_{s'}C_{t'}^\lambda\right)b_{j'} \bmod \hat{A}^{(D,\lambda)} \\
&= \left\langle \phi_j(X), r_{m(i,j)} \cdot \phi_i(Y)\right\rangle_D \cdot {}_{(R_{i'},s')}C^{(D,\lambda)}_{(L_{j'},t')} \bmod \hat{A}^{(D,\lambda)}.
\end{aligned}$$

This gives $\langle X, Y\rangle = \left\langle \phi_j(X), r_{m(i,j)}\phi_i\left(Y\right)\right\rangle_D = \left\langle r_{m(i,j)}\phi_j(X), \phi_i\left(Y\right)\right\rangle_D$, proving part (b). $\square$

We note the following corollary for future use.

Corollary 4.1. *Let M be a finite monoid satisfying the R-C.A. condition and place the standard cell algebra structure on $R[M]$. Then for any $\lambda \in \Lambda_D$, $D \in \mathbb{D}$:*

1. *If* $\mathrm{rad}_D\left({}_LC^\lambda\right) \neq 0$, *then* $\mathrm{rad}\left({}_LC^{(D,\lambda)}\right) \neq 0$,

2. *If* $\mathrm{rad}_D\left(D_R^\lambda\right) \neq 0$, *then* $\mathrm{rad}\left(D_R^{(D,\lambda)}\right) \neq 0$.

Proof. Assume $\mathrm{rad}_D\left({}_LC^\lambda\right) \neq 0$ and take a $y \neq 0$ in $\mathrm{rad}_D\left({}_LC^\lambda\right)$. Write $y = \sum_{s\in L(\lambda)} c(s) \cdot {}_sC^\lambda$ and put $Y = \sum_{s\in L(\lambda)} c(s)\, {}_{(R_i,s)}C^{(D,\lambda)} \in \left({}_LC^{(D,\lambda)}\right)_i \subseteq {}_LC^{(D,\lambda)}$ for some $R_i \subseteq D$. Then $Y \neq 0$ and we claim $Y \in \mathrm{rad}\left({}_LC^{(D,\lambda)}\right)$: Take any $X \in \left(C_R^{(D,\lambda)}\right)_j$. Then by the proposition, if R_i, L_j are not matched we have $\langle X, Y\rangle = 0$, while if R_i, L_j are matched, then

$$\langle X, Y\rangle = \left\langle r_{m(i,j)}\phi_j(X), \phi_i(Y)\right\rangle_D = \left\langle r_{m(i,j)}\phi_j(X), y\right\rangle_D = 0$$

since $y \in \mathrm{rad}_D\left({}_LC^\lambda\right)$. Then $\langle X, Y\rangle = 0$ for any $X \in C_R^{(D,\lambda)}$ and $Y \in \mathrm{rad}\left({}_LC^{(D,\lambda)}\right)$ as claimed. The proof of ii. is parallel. $\square$

Write $D^2 = \{xy : x, y \in D\}$ and recall that $\hat{A}^D = \oplus_{D'<D} R[D'] \subseteq \hat{A}^{(D,\lambda)}$ for any $\lambda \in \Lambda_D$.

Corollary 4.2. *For any $\mathcal{D}$-class D,*

(a) *If $D^2 \subseteq \hat{A}^D$, then $(D,\lambda) \notin \Lambda_0$ for any $\lambda \in \Lambda_D$*

(b) *If $D^2 \not\subseteq \hat{A}^D$, then $(D,\lambda) \in \Lambda_0 \Leftrightarrow \lambda \in (\Lambda_D)_0$.*

Proof. $D^2 \subseteq \hat{A}^D$ if and only if no pair R_i, L_j are matched.

(a) $D^2 \subseteq \hat{A}^D$ implies no pair R_i, L_j is matched, so by proposition 4.1 $\langle X, Y\rangle = 0$ for every $X \in C_R^{(D,\lambda)}, Y \in {}_LC^{(D,\lambda)}$. Then $\mathrm{rad}\left(C_R^{(D,\lambda)}\right) = C_R^{(D,\lambda)}$ and $(D,\lambda) \notin \Lambda_0$.

(b) If $D^2 \not\subseteq \hat{A}^D$ then R_i, L_j are matched for at least one pair i, j.

For $\Rightarrow$: If $(D,\lambda) \in \Lambda_0$, then $\langle X, Y\rangle \neq 0$ for some $X \in \left(C_R^{(D,\lambda)}\right)_j$, $Y \in \left({}_LC^{(D,\lambda)}\right)_i$, where i, j must be matched. Then by proposition 4.1, $\langle X, Y\rangle = \left\langle \phi_j(X), r_{m(i,j)}\phi_i(Y)\right\rangle_D \neq 0$, where $\phi_j(X) \in C_R^\lambda$, $r_{m(i,j)}\phi_i(Y) \in {}_LC^\lambda$. So $\mathrm{rad}_D\left(C_R^\lambda\right) \neq C_R^\lambda$ and $\lambda \in (\Lambda_D)_0$.

For $\Leftarrow$: If $\lambda \in (\Lambda_D)_0$, then there exist $\xi \in C_R^\lambda$, $\eta \in {}_LC^\lambda$ with $\langle \xi, \eta\rangle_D \neq 0$. Choose i, j with R_i, L_j matched and let $X = \phi_j^{-1}(\xi) \in \left(C_R^{(D,\lambda)}\right)_j$, $Y = \phi_i^{-1}(r_{m(i,j)}^{-1} \cdot \eta) \in \left({}_LC^{(D,\lambda)}\right)_i$. (Note that $r_{m(i,j)} \in G_D$, so $r_{m(i,j)}^{-1} \in G_D$ is well defined.) Then by proposition 4.1, $\langle X, Y\rangle = \left\langle \phi_j(X), r_{m(i,j)}\phi_i(Y)\right\rangle_D = \langle \xi, \eta\rangle_D \neq 0$. Then $\mathrm{rad}\left(C_R^{(D,\lambda)}\right) \neq C_R^{(D,\lambda)}$ and $(D,\lambda) \in \Lambda_0$. $\square$

Consider the question of whether a monoid algebra $k[M]$ is quasi-hereditary (as defined by [2]). The following was proved in [11]:

Proposition 4.2. *Let k be a field and A be a cell algebra over k such that $\Lambda_0 = \Lambda$. Then A is quasi-hereditary.*

Then by corollary 4.2 we have

Corollary 4.3. *Let k be a field and M a finite monoid such that for every class $D \in \mathbb{D}$, $D^2 \not\subset \hat{A}^D$ and the group algebra $k[G_D]$ of the Schutzenberger group for D is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A = k[M]$ is quasi-hereditary.*

Recall that a semigroup M is *regular* if for every $x \in M$ there is a $y \in M$ such that $x = xyx$. In a regular semigroup, every $\mathcal{D}$-class D contains an idempotent ρ. Then $\rho = \rho^2 \in D \cap D^2$, so $D^2 \not\subset \hat{A}^D$. Thus

Corollary 4.4. *Let k be a field and M a regular finite monoid such that for every class $D \in \mathbb{D}$, the group algebra $k[G_D]$ of the Schutzenberger group for D is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A = k[M]$ is quasi-hereditary.*

Note that in a regular semi-group the base $\mathcal{H}$-class H for any class D can be chosen to contain an idempotent. Then H is a (maximal) subgroup of M with H isomorphic to G_D. So the condition in corollary 4.4 can be replaced by the requirement that for any maximal subgroup G of M the group algebra $k[G]$ must be a cell algebra with $(\Lambda_G)_0 = \Lambda_G$.

As a special case we obtain the following result of Putcha [13].

Corollary 4.5. *For any regular finite monoid M, the complex monoid algebra $A = \mathbb{C}[M]$ is quasi-hereditary.*

Proof. As remarked above, each $\mathbb{C}[G_D]$ will be (split) semisimple by Maschke's theorem. As a product of complex matrix algebras it has a natural cellular basis and (being semisimple) is quasi-hereditary. Each $\mathbb{C}[G_D]$ then satisfies the conditions of 4.4, so A is quasi-hereditary. $\square$

Now consider the question of semisimplicity for a monoid algebra $k[M]$. The following criterion for semisimplicity of a cell algebra follows from results in [11].

Proposition 4.3. *A cell algebra A over a field k is semisimple if and only if for every $\lambda \in \Lambda$ we have ${}_LC^\lambda = {}_LD^\lambda$ and $C_R^\lambda = D_R^\lambda$.*

Proof. Assume A is semisimple. Then for every $\lambda \in \Lambda_0$ we have ${}_LP^\lambda = {}_LD^\lambda$, where ${}_LP^\lambda$ is the principle indecomposable left module corresponding to the irreducible ${}_LD^\lambda$. But since ${}_LP^\lambda$ always has a filtration with ${}_LC^\lambda$ as the "top" quotient, we must have ${}_LP^\lambda = {}_LC^\lambda = {}_LD^\lambda$. Since each ${}_LD^\lambda$ is absolutely irreducible, the multiplicity of ${}_LP^\lambda$ as a direct summand in A is just $\dim\left({}_LD^\lambda\right) = \dim\left({}_LC^\lambda\right) = |L(\lambda)|$. So $\dim(A) = \sum_{\lambda\in\Lambda_0} |L(\lambda)|^2$. Similarly, for every $\lambda \in \Lambda_0$ we have $P_R^\lambda = C_R^\lambda = D_R^\lambda$ (where P_R^λ is the principle indecomposable right module corresponding to the irreducible D_R^λ) and $\dim(A) = \sum_{\lambda\in\Lambda_0} |R(\lambda)|^2$. But the multiplicity of ${}_LP^\lambda$ as a direct summand in A and the multiplicity of P_R^λ as a direct summand in A must be equal (as both equal the number of primitive idempotents corresponding to $\lambda \in \Lambda_0$). So $|L(\lambda)| = |R(\lambda)|$ and we can write $\dim(A) = \sum_{\lambda\in\Lambda_0} |L(\lambda)| \cdot |R(\lambda)|$. On the other hand, a direct count of basis elements ${}_sC_t^\lambda$, $\lambda \in \Lambda$, $s \in L(\lambda)$, $t \in R(\lambda)$, shows that $\dim(A) = \sum_{\lambda\in\Lambda} |L(\lambda)| \cdot |R(\lambda)|$. It follows that $\Lambda_0 = \Lambda$, so ${}_LC^\lambda = {}_LD^\lambda$ and $C_R^\lambda = D_R^\lambda$ for every $\lambda \in \Lambda$.

Now assume ${}_LC^\lambda = {}_LD^\lambda$ and $C_R^\lambda = D_R^\lambda$ for every $\lambda \in \Lambda$, so $\Lambda_0 = \Lambda$. As in [11], for $\mu \in \Lambda_0$, $\lambda \in \Lambda$, let $Rd_{\lambda\mu} = [C_R^\lambda : D_R^\mu]$ be the multiplicity of the irreducible D_R^μ as a composition factor in C_R^λ, let $Ld_{\lambda\mu} = \left[{}_LC^\lambda : {}_LD^\mu\right]$ be the multiplicity of ${}_LD^\mu$ in ${}_LC^\lambda$, and write $RD = (Rd_{\lambda\mu})$, $LD = (Ld_{\lambda\mu})$ for the decomposition matrices of A. Then RD and LD are both square identity matrices. The right Cantor matrix, RC, and left Cantor matrix, LC, are the square $|\Lambda_0| \times |\Lambda_0|$ matrices where for $\lambda, \mu \in \Lambda_0$, $RC_{\lambda\mu} = \left[P_R^\lambda : D_R^\mu\right]$ and $LC_{\lambda\mu} = \left[{}_LP^\lambda : {}_LD^\mu\right]$. As shown in [11], we have $RC = LD^T \cdot RD$ and $LC = RD^T \cdot LD$ (where T denotes the transpose matrix). Then RC and LC are also identity matrices and we must have $P_R^\lambda = D_R^\lambda$ and ${}_LP^\lambda = {}_LD^\lambda$ for every $\lambda \in \Lambda = \Lambda_0$. Since A is a direct sum of principle indecomposable left modules isomorphic to the various ${}_LP^\lambda = {}_LD^\lambda$, it is a direct sum of irreducible modules and therefore semisimple. $\square$

Consider a finite monoid M satisfying the R-C.A. condition and place the standard cell algebra structure on M.

Corollary 4.6. *If $k[M]$ is semisimple, then $k[G_D]$ is semisimple for every $D \in \mathbb{D}$.*

Proof. If $k[G_D]$ is not semisimple for some D, then by the proposition either $\mathrm{rad}_D\left({}_LC^\lambda\right) \neq 0$ or $\mathrm{rad}_D\left(D_R^\lambda\right) \neq 0$. Assume $\mathrm{rad}_D\left({}_LC^\lambda\right) \neq 0$ (if $\mathrm{rad}_D\left(D_R^\lambda\right) \neq 0$ the proof is similar). Then by corollary 4.1 we have $\mathrm{rad}\left({}_LC^{(D,\lambda)}\right) \neq 0$. But then $k[M]$ is not semisimple by proposition 4.3. $\square$

If all the algebras $k[G_D]$ are semisimple, we would like a condition guaranteeing that $k[M]$ itself is semisimple. For any $\mathcal{D}$-class D, let $\mathbb{L}(D)$ be the set of all $\mathcal{L}$-classes contained in D and $\mathbb{R}(D)$ be the set of all $\mathcal{R}$-classes contained in D. Define the "bijection condition" on D:

Bijection condition: There exists a bijection $F : \mathbb{L}(D) \to \mathbb{R}(D)$ such that L and $F(L)$ are matched while L and R are not matched if $R \neq F(L)$.

Proposition 4.4. *Let k be a field and M a monoid such that for every $\mathcal{D}$-class D*

1. *$k[G_D]$ is a semisimple cell algebra and*
2. *the bijection condition is satisfied for D.*

Then $k[M]$ is a semisimple cell algebra.

Proof. Place the standard cell algebra structure on $k[M]$ as in theorem 3.1. We will use proposition 4.3 to show semisimplicity. Take $(D, \lambda) \in \Lambda$. We will show $\mathrm{rad}\left({}_LC^{(D,\lambda)}\right) = 0$ and therefore ${}_LC^{(D,\lambda)} = {}_LD^{(D,\lambda)}$. Take any $Y = \sum_k Y_k \in \mathrm{rad}\left({}_LC^{(D,\lambda)}\right)$ where $Y_k \in \left({}_LC^{(D,\lambda)}\right)_k$. Then $\langle X, Y \rangle = 0$ for any $X \in C_R^{(D,\lambda)}$. If $X \in \left(C_R^{(D,\lambda)}\right)_j$, then $\langle X, Y_k \rangle = 0$ whenever $L_j \neq F(R_k)$ by proposition 4.1. Then we must also have $\langle X, Y_k \rangle = 0$ when $L_j = F(R_k)$. But when L_j, R_k are matched, proposition 4.1 gives $\langle X, Y_k \rangle = \left\langle \phi_j(X), r_{m(i,j)}\phi_i\left(Y_k\right)\right\rangle_D = 0$. Since

$\phi_j(X)$ is an arbitrary basis element in C_R^λ, we have $r_{m(i,j)}\phi_i(Y_k) \in \mathrm{rad}_D\left({}_LC^\lambda\right)$. But since $k[G_D]$ is semisimple, $\mathrm{rad}_D\left({}_LC^\lambda\right) = 0$ by proposition 4.3. Then $r_{m(i,j)}\phi_i(Y_k) = 0$, and since $r_{m(i,j)} \in G_D$ is invertible, $Y_k = 0$. Since this is true for each k, we have $Y = 0$. So $\mathrm{rad}\left({}_LC^{(D,\lambda)}\right) = 0$. A parallel argument shows that $\mathrm{rad}\left(C_R^{(D,\lambda)}\right) = 0$, so $k[M]$ is semisimple by proposition 4.3. □

Recall that an *inverse* of an element a in a semi-group is an element a^{-1} such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. An *inverse semi-group* is a semi-group in which each element has a unique inverse. A standard result in semi-group theory is that any inverse semi-group satisfies the bijection condition. (In fact each $\mathcal{L}$-class and each $\mathcal{R}$-class contains a unique idempotent. Given an $\mathcal{L}$-class L, the unique matching $\mathcal{R}$-class $R = F(L)$ is the class containing the same idempotent as L.) Then proposition 4.4 and corollary 4.6 yield

Corollary 4.7. *Let k be a field and M a finite monoid which is an inverse semi-group and satisfies the k-C.A. condition. Then $k[M]$ is semisimple if and only if $k[G_D]$ is semisimple for every $\mathcal{D}$-class D.*

For an inverse semi-group the base $\mathcal{H}$-class H for any class D can be chosen to be a (maximal) subgroup of M with H isomorphic to G_D. So the condition in corollary 4.7 can be replaced by the requirement that for any maximal subgroup G of M the group algebra $k[G]$ must be a semisimple cell algebra.

It is well known that if the finite monoid M is an inverse semi-group and the field k has characteristic not dividing the order of any G_D, then $k[M]$ is semisimple (see e.g. [1], which cites [12]). In this case each $k[G_D]$ is semisimple by Maschke's theorem. If k is also algebraically closed, then as remarked above, M satisfies the k-C.A. condition. Thus corollary 4.7 yields the semisimplicity of $k[M]$ under the additional hypothesis of algebraic closure for k.

5 Twisted monoid algebras

A *twisting* on a monoid M (with values in a commutative domain R with unit 1) is a map $\pi : M \times M \to R$ such that (i) for all $x, y, z \in M$,

$$\pi(x,y)\,\pi(xy,z) = \pi(x,yz)\,\pi(y,z)$$

and (ii) for all $x \in M$,

$$\pi(x, id) = 1 = \pi(id, x)$$

(where id is the identity in M).

Given a twisting π on M, define an algebra $R^\pi[M]$ to be the free R-module with basis M and multiplication $x \circ y = \pi(x,y)\,xy$ for $x, y \in M$. Then $R^\pi[M]$ is an associative R-algebra with unit 1.

We would like conditions under which $R^\pi[M]$ will be a cell algebra. In [14] and [9], Wilcox and Guo and Xi have investigated when $R^\pi[M]$ can be a cellular algebra. Much of the difficulty in their analyses involves defining the involution

anti-isomorphism $*$ required for a cellular algebra. Since cell algebras don't require such a map, the corresponding results are both simpler to obtain and of more general applicability. We require one "compatibility" condition for our twisting:

Definition 5.1. *A twisting π on a monoid M is* compatible *if for all $a, x \in M$*

1. *If $ax\mathcal{D}x$, then $\pi(a, y) = \pi(a, x)$ whenever $y \in H_x$,*
2. *If $xa\mathcal{D}x$, then $\pi(y, a) = \pi(x, a)$ whenever $y \in H_x$.*

In [7], π is defined to be an $\mathcal{LR}$-twisting if $x\mathcal{L}y \Rightarrow \pi(x, z) = \pi(y, z)$ and $y\mathcal{R}z \Rightarrow \pi(x, y) = \pi(x, z)$ for all $x, y, z \in M$. Clearly any $\mathcal{LR}$-twisting is compatible.

Theorem 5.1. *Let M be a finite monoid satisfying the R-C.A. condition. Let π be a compatible twisting on M. Then $R^\pi[M]$ has a cell algebra structure with the same Λ, R, L and cell basis C as for the standard cell algebra structure on $R[M]$.*

Proof. Since as R-modules $R^\pi[M]$ and $R[M]$ are identical, C is an R-basis for $R^\pi[M]$ and we need only check that conditions (i) and (ii) for a cell algebra are satisfied for the new multiplication.

For (i), assume a is a basis element, $a \in M$, and take any ${}_{(R,s)}C^{(D,\lambda)}_{(L,t)} \in C$. Then ${}_{(R,s)}C^{(D,\lambda)}_{(L,t)} \in R[H]$ is an R-linear combination of elements in an $\mathcal{H}$-class $H = L \cap R \subseteq D$. By corollary 2.2 we have two possibilities: I. $a \cdot R[H] \subseteq \hat{A}^D = \oplus_{D' < D} R[D']$ or II. $a \cdot R[H] \subseteq D$. If I. holds, $a \cdot {}_{(R,s)}C^{(D,\lambda)}_{(L,t)} \subseteq \hat{A}^D \subseteq \hat{A}^{(D,\lambda)}$ and we can take the coefficients r_L required in (i) to be all 0. So assume II. holds. Take an element $x \in H$ and let $c = \pi(a, x)$. Since $ax \in D = D_x$, compatibility gives $\pi(a, y) = c$ for any $y \in H$. But then, by linearity, $a \circ {}_{(R,s)}C^{(D,\lambda)}_{(L,t)} = c \cdot a \cdot {}_{(R,s)}C^{(D,\lambda)}_{(L,t)}$. We can then take the coefficients r_L required in (i) to be just c times the corresponding coefficients in the cell algebra $R[M]$.

The proof of condition (ii) is parallel. ∎

Let M be a finite monoid satisfying the R-C.A. condition and place the standard cell algebra structure on M. Suppose R_i , L_j are matched classes in a $\mathcal{D}$-class D of M. Then by definition there exist $x \in L_j$, $y \in R_i$ such that $xy \in D$. If π is a compatible twisting on M define $c(j, i) = \pi(x, y)$. Then $\pi(a, b) = c(j, i)$ for any $a \in L_j$, $b \in R_i$, so for any $X \in R[L_j]$, $Y \in R[R_i]$ we have $X \circ Y = c(j, i)XY$. It follows that for $X \in \left(C_R^{(D,\lambda)}\right)_j$, $Y \in \left({}_LC^{(D,\lambda)}\right)_i$ we have $\langle X, Y\rangle^\pi = c(j, i) \langle X, Y\rangle$ where $\langle X, Y\rangle^\pi$ is the bracket in the cell algebra $R^\pi[M]$.

Definition 5.2. *A twisting π on a monoid M is* strongly compatible *if for all $a, x \in M$*

1. *If $ax\mathcal{D}x$, then $\pi(a, y) = \pi(a, x) \neq 0$ whenever $y \in H_x$,*

2. *If $xa\mathcal{D}x$, then $\pi(y,a) = \pi(x,a) \neq 0$ whenever $y \in H_x$.*

So for a strongly compatible twisting we have $c(j,i) \neq 0$ whenever R_i , L_j are matched classes. But $c(j,i) \neq 0$ in the domain R yields $\langle X,Y\rangle^{\pi} = 0 \Leftrightarrow \langle X,Y\rangle = 0$. Using this observation, it is easy to modify the proofs to obtain the following generalizations to twisted monoid algebras of the results in section 4.

In the following, assume R is a domain, M a finite monoid satisfying the R-C.A. condition, and π a strongly compatible twisting from M to π. Put the standard cell algebra structures on $R[M]$ and $R^{\pi}[M]$ as given by theorems 3.1 and 5.1. Let $\Lambda, \Lambda_0, \langle -,-\rangle$, rad, etc., refer to the cell algebra $R[M]$ and $\Lambda^{\pi}, \Lambda_0^{\pi}, \langle -,-\rangle^{\pi}$ rad$^{\pi}$, etc., refer to the cell algebra $R^{\pi}(M)$. Recall that $\Lambda^{\pi} = \Lambda$.

Proposition 5.1. *For any $\lambda \in \Lambda_D$, $D \in \mathbb{D}$:*

1. *If $\mathrm{rad}_D\left({}_LC^{\lambda}\right) \neq 0$, then $\mathrm{rad}^{\pi}\left({}_LC^{(D,\lambda)}\right) \neq 0$,*

2. *If $\mathrm{rad}_D\left(D_R^{\lambda}\right) \neq 0$, then $\mathrm{rad}^{\pi}\left(D_R^{(D,\lambda)}\right) \neq 0$.*

Proposition 5.2. *For any $\mathcal{D}$-class D,*

(a) *If $D^2 \subseteq \hat{A}^D$, then $(D,\lambda) \notin \Lambda_0^{\pi}$ for any $\lambda \in \Lambda_D$*

(b) *If $D^2 \not\subset \hat{A}^D$, then $(D,\lambda) \in \Lambda_0^{\pi} \Leftrightarrow \lambda \in (\Lambda_D)_0$.*

Proposition 5.3. *Let $R = k$ be a field. Assume that for every class $D \in \mathbb{D}$, $D^2 \not\subset \hat{A}^D$ and $k[G_D]$ is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A^{\pi} = k^{\pi}[M]$ is quasi-hereditary.*

Proposition 5.4. *Let $R = k$ be a field. Assume that M is regular and that for every class $D \in \mathbb{D}$, $k[G_D]$ is a cell algebra with $(\Lambda_D)_0 = \Lambda_D$. Then $A^{\pi} = k^{\pi}[M]$ is quasi-hereditary.*

Proposition 5.5. *If $R = k$ is a field and $k^{\pi}[M]$ is semi-simple, then $k[G_D]$ is semi-simple for every $D \in \mathbb{D}$.*

Proposition 5.6. *Let $R = k$ be a field. Assume that for every $\mathcal{D}$-class D*

1. *the cell algebra $k[G_D]$ is a semi-simple and*

2. *the bijection condition is satisfied for D.*

Then the cell algebra $k^{\pi}[M]$ is semi-simple.

Proposition 5.7. *Let $R = k$ be a field. Assume the monoid M is also an inverse semi-group. Then $k^{\pi}[M]$ is semi-simple if and only if $k[G_D]$ is semi-simple for every $\mathcal{D}$-class D.*

The various examples such as Brauer algebras, Temperly-Lieb algebras, and other parition algebras which were studied and shown to be cellular in [14] and [9] are all twisted monoid algebras with a compatible twisting on a monoid satisfying the R-C.A. condition. Thus they can be seen to be cell algebras by

Theorem 5.1 without constructing the anti-isomorphism needed for the cellular structure. Related algebras which lack the anti-isomorphism $*$, and hence are not cellular, could also be shown to be cell algebras by Theorem 5.1. We note again that questions such as whether an algebra is quasi-hereditary or semisimple are not much harder to answer for cell algebras than for cellular algebras.

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